Numbers and Knots are Fun

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1 Even Perfect Numbers

Definition 1. A perfect number is a whole number whose divisors (including itself) sum to twice that number.

Conjecture 1. *There exist an infinite number of perfect numbers.*

An example of a perfect number is 28 : It's divisors are $\{1, 2, 4, 7, 14, 28\}$ and $1 + 2 + 4 + 7 + 14 + 28 = 56 = 28 \times 2$. The first six perfect numbers are $\{6, 28, 496, 8128, 33550336, 8589869056\}$.

So far, only even perfect numbers have been discovered and the existence of infinite even perfect is known to be contingent on the existence of an infinite number of Mersenne primes (see Theorems 1 and 2). So far, 51 perfect numbers are known. It is unknown if there are any odd perfect numbers.

Conjecture 2. *There exist no odd perfect numbers*

Definition 2. A *Mersenne prime* is a prime number of the form $2^p - 1$.

Definition 3. The sum of all the divisors of a natural number n, including n iteself, is denoted $\sigma(n)$. A natural number n is *deficient* if $\sigma(n) < 2n$. As previously stated, a natural number n is *perfect* if $\sigma(n) = 2n$. A natural number n is *abundant* if $\sigma(n) > 2n$.

1.1 The Form of Even Perfect Numbers

Prove: all even perfect numbers are of the form

$$2^{p-1}(2^p - 1)$$

where $2^p - 1$ is prime.

Lemma 1. $2^p - 1$ *is prime if and only if* p *is prime.*

Proof. Assume on the contrary $2^p - 1$ is prime but p is composite.
Thus $\exists r, s \in N, 1 < r < s < p$
such $rs = p$

Therefore

$$2^p - 1 = (2^r)^s - 1^s = (2^r - 1)((2^r)^{s-1} + (2^r)^{s-2} + \cdots + 2^r + 1).$$

a contradiction □

Theorem 1. *if* $2^p - 1$ *is prime, then* $2^{p-1}(2^p - 1)$ *is perfect*

Proof. If $2^p - 1$ is prime,
then $\sigma(2^{p-1}(2^p - 1)) = 2^p(2^p - 1)$
consider $S = \{\forall s \in \mathbb{Z}^+ , s|2^{p-1}(2^p - 1)|1, 2, 2^2, \ldots, 2^{p-1}, 2^p - 1, 2(2^p - 1), 2^2(2^p - 1), \ldots, 2^{p-1}(2^p - 1)\}$

$$\sum(S) = \sigma(2^{p-1}(2^{p-1})$$

$$\sigma(2^{p-1}(2^p - 1)) = (1 + 2 + 2^2 + \cdots + 2^{p-1})(1 + (2^p - 1)) = (2^p - 1)(2^p)$$

□

Theorem 2. *If* $2^{p-1}(2^p - 1)$ *is perfect,*
then $2^p - 1$ *is prime.*

Proof. Prove: If $\sigma(2^k x) = 2^{k+1}x$, $2 \nmid x$, $k, x \in \mathbb{Z}^+$
then x is a Mersenne prime of form $2^{k+1} - 1$.
Assume on the contrary that x is not of the form $2^{k+1} - 1$ yet

$$\sigma(2^k \times x) = 2^{k+1} \times x$$

.
Because the σ function is multiplicative it is known:
$\sigma(2^{k+1}x) = \sigma(2^k)\sigma(x)$

$$\sigma(2^k) = 1 + 2 + 2^2 + \cdots + 2^k = 2^{k+1} - 1$$

because $gcd(2^{k+1} - 1, 2^k) = 1$,
it is known $2^{k+1} - 1|x$
thus $\exists y \in \mathbb{Z}^+, y = x/(2^{k+1} - 1)$ such

$$2^{k+1}y = \sigma(x)$$

since we known x has at least two divisors (x and y both divide x) we can write

$$\sigma(x) = x + y + others$$

$$\sigma(x) = y(2^{k+1} - 1) + y + others = 2^{k+1}y$$

$$others = 0$$

Thus x is prime , $x = 2^{k+1} - 1, y = 1$ □

Therefore, all even perfect numbers are of the form $2^{p-1}(2^p - 1)$ and the existence of an infinite number of even perfect numbers is **contingent** on the existence of an infinite number of Mersenne primes.

1.2 Sums of Digits

Theorem 3. *Repeated sums of digits of perfect numbers always end up equalling to 1*

Proof. We will prove by using different Lemmas and then combining their results together

Lemma 2. *For any integer n,* $S(n) = n \pmod 9$ *where S(n) is the function for repeated sum of digits of n*

Proof. Expand n where a represents its ones digit, tens digit, etc.

$$n = a_i 10^{+} \cdots + a_2 10^2 + a_1 10^1 + a_0 10^0$$

$$n = a_i (99...9 + 1) + ... + a_2 (99 + 1) + a_1 (9 + 1) + a_0 (0 + 1)$$

$$n = 9 [a_i (11_{...}1) + ... + a_2 (11) + a_1 (1) + a_0 (0)] + \sum_{j=0}^{i} a_j$$

$$n \pmod 9 = \sum_{j=0}^{i} a_j$$

□

Lemma 3. *All even perfect numbers are in the form* $2^{p-1} (2^p - 1)$ *where p is prime.*

Proof. Check earlier pages for the proof □

Lemma 4. *All primes are in the form 1 (mod 6) or 5 (mod 6) except for 2 and 3*

Proof. Prove using casework where x and y are integers:
Case 1 -

$$let\ x = y (6) + 2 = 2 (3y + 1)$$

Therefore x is not a prime
Case 2 -

$$let\ x = y (6) + 3 = 3 (2y + 1)$$

Therefore x is not a prime
Case 3 -

$$let\ x = y (6) + 4 = 2 (3y + 2)$$

Therefore x is not a prime
This means that a prime number must be in the form 1 (mod 6) or 5 (mod 6) because it is not able to be factored unlike cases 1, 2, and 3 □

Now we can prove the theorem with the lemmas.

From Lemma 1.5 we know the form of prime numbers and therefore can rewrite p (a prime number) where m is an integer:

$$let\ p = 6m \pm 1$$

We will only provide the example of when $p = 6m+1$ but the process is exactly the same for $p = 6m - 1$.
From Lemma 1.4, we can substitute the value of p.

$$\left[2^{(6m+1)-1}\left(2^{6m+1}-1\right)\right] \pmod 9$$

$$= 2^{6m} \pmod 9 \cdot \left(2^{6m+1}-1\right) \pmod 9$$

Powers of 2 always cycle periods of 6 and cycle the result of 1(mod 9). This can be easily proved by writing out the powers of 2 and recognizing the pattern. This can be summarized by this formula: $2^{6m} \equiv 1 \pmod 9$
Now substitute this fact into the previous equation:

$$= 1 \pmod 9 \cdot \left[2^{6m+1} \pmod 9 - 1 \pmod 9\right]$$

$$= 1 \pmod 9 \cdot \left[2^{6m} \pmod 9 \cdot 2 \pmod 9\right] - 1 \pmod 9$$

$$= 1 \pmod 9 \cdot [2 \pmod 9 - 1 \pmod 9]$$

$$= 1 \pmod 9$$

Therefore from the formula in Lemma 1.3, we have proved that repeated sums of digits of perfect numbers equal 1 □

1.3 NIFTY PROOFS

Theorem 4. *All even perfect numbers are triangular numbers*

Proof. Definition: a triangular number, T_k, is of the form

$$k \in \mathbb{Z}^+, T_k = k(k+1)/2$$

Each triangular number T_k equals the sum of the first k natural numbers

Fact: Every even perfect number, P is of the form $P = 2^{p-1}(2^p - 1)$, where $2^p - 1$ is prime

$$P = 2^p(2^p - 1)/2$$

□

Theorem 5. *All perfect numbers have a last digit 6 or 8*

Proof. Prove: All even perfect numbers have a last digit 6 or 8

Fact 1: $6^k \cong 6(mod10), k \in N$

Fact 2: $4^k \cong 4(mod10)$ $2 \nmid k$, $4^k \cong 6(mod10)$ if 2|k

$$P = 2^{p-1}(2^p - 1), \text{ where } 2^p - 1, p \text{ are both prime}$$

since p is prime , $p = 4n + 1 or p = 4n + 3$

Case One: $p = 4n + 1$

$$2^{4n+1-1}(2^{4n+1}-1) = 16^n(2\times16^n-1) \cong 6^n(2\times6^n-1)(mod10) \cong 6(2\times6-1)(mod10) \cong 66(mod10) \cong 6(mod10)$$

Case Two: $p = 4n + 3$

$$2^{4n+3-1}(2^{4n+3}) = 4^{2n+1}(8\times16^n-1) \cong 4(8\times6^n-1)(mod10) \cong 4(47)(mod10) \cong 8(mod10)$$

□

Theorem 6. *All even perfect numbers are in the form of the sum of consecutive odd cubes*

We will first need to prove the following lemma

Lemma 5. *Consecutive sum of cubes can be written in the form* $\frac{n^2(n+1)^2}{4}$ *where* n *= number of consecutive integers*

Proof. Prove using induction Induction Hypothesis: Assume $1^3 + 2^3 + ... + n^3 = \frac{n^2(n+1)^2}{4}$ holds true for all values of n

Base Case:

$$Let\, n\ = 1\ -\ LHS\ =\ 1,\ RHS\ =\ \frac{1\left(2^2\right)}{4} = 1$$

Induction Step: Let the induction hypothesis hold true when n=k

We need to prove $1^3 + 2^3 + ... + (k+1)^3 = \frac{(k+1)^2(k+2)^2}{4}$

By induction hypothesis we know that $1^3 + 2^3 + ... + k^3 = \frac{k^2(k+1)^2}{4}$

Therefore when adding $(k+1)^3$ to both sides of the equation:

$$1^3 + 2^3 + ... + k^3 + (k+1)^3 = \frac{k^2(k+1)^2}{4} + (k+1)^3$$

$$= \frac{k^2(k+1)^2 + 4(k+1)^3}{4}$$

$$= \frac{(k+1)^2\left(k^2 + 4k + 4\right)}{4}$$

$$= \frac{(k+1)^2(k+2)^2}{4}$$

□

Now we can prove the theorem:

Proof. Let m=$2^{\frac{p-1}{2}}$ where p is prime We want to prove that:

$$1^3 + 3^3 + ... + (2m-1)^3 = 2^{p-1}\left(2^p - 1\right)$$

Rewrite the LHS:

$$LHS = \left(1^3 + 2^3 + 3^3 + ... + (2m)^3\right) - \left(2^3 + 4^3 + ... + (2m)^3\right)$$

$$= \left(1^3 + 2^3 + 3^3 + ... + (2m)^3\right) - 2^3\left(1^3 + 2^3 + 3^3 + ... + m^3\right)$$

Using Lemma 1.8 we can simplify the equation:

$$= \frac{(2m)^2 (2m+1)^2}{4} - 2^3\left(\frac{m^2 (m+1)^2}{4}\right)$$

$$= m^2 (2m+1)^2 - 2m^2 (m+1)^2$$

$$= m^2\left(4m^2 + 4m + 1 - 2m^2 - 4m - 2\right)$$

$$= m^2\left(2m^2 - 1\right)$$

Substitute m=$2^{\frac{p-1}{2}}$

$$= 2^{p-1}\left(2\left(2^{p-1}\right) - 1\right)$$

$$= 2^{p-1}\left(2^p - 1\right)$$

□

1.4 Lucas-Lehmer Primality Test

Theorem 7. $2^p - 1$ *is prime iff* $2^p - 1 | S_{p-2}, S_p = S_{p-1}^2 - 2S_0 = 4$

Lemma 6. $\forall x \in groupG, |x| \leq |G|$ *where x is not the identity element in G*

Proof. Let $G = \{1, a_1, a_2, ..., a_q\} |G| = q + 1, x = a_i, i \in \mathbb{N}$,$1 \leq i \leq q$

Assume on the contrary $|x| > |G|$

Thus $|x| > q + 1$

Therefore $\nexists m, n \in \mathbb{N}$, $m \neq n$ such $x^m = x^n$

because if $x^m = x^n$

then there exists some $n - m \in \mathbb{N} 1 \leq n - m < |x|$ such $x^{n-m} = 1$ a contradiction.

$$\forall r,\ 1 \leq r \leq q, x^r \text{ is in one to one correspondence to a particular } a_i$$

since it is known

$$x^q \neq x^{q+1}, x^{q+1} = 1,$$

a contradiction.

□

Lemma 7. $\forall x \in$ *group* G *and* $p \in \mathbb{N}$
if $x^p = 1,$ *then* $|x||p$

Proof. Let $x \in$ Group $G,$ and x has finite order

Assume on the Contrary $x^p = 1|x| < p$ but $|x| \nmid p$ Thus

$$\exists r \in N1 \leq r \leq |x| - 1$$

such $p = q|x| + r$

$$1 = x^p = (x^{|x|})^q \times x^r = 1$$

$$x^r = 1$$

Because |x| is the order of x, it represents the smallest power m such $x^m = 1$ since r<|x| and $x^r = 1$ There is a contradiction

□

Let $S_p = S_{p-1}^2 - 2, S_0 = 4, \omega = 2 + \sqrt{3}\bar{\omega} = 2 - \sqrt{3}\omega \times \bar{\omega} = 1$
Prove:

$$S_p = \omega^{2^p} + \bar{\omega}^{2^p}$$

Base Case: $p = 0, S_0 = 4, \omega^{2^0} + \bar{\omega}^{2^0} = 2 + \sqrt{3} + 2 - \sqrt{3} = 4 = 4$
Inductive hypothesis: If $\omega^{2^{k+1}} + \bar{\omega}^{2^{k+1}} = (\omega^{2^k} + \bar{\omega}^{2^k})^2 - 2$
Then

$$S_p = \omega^{2^p} + \bar{\omega}^{2^p}$$

$$(\omega^{2^k} + \bar{\omega}^{2^k})^2 - 2 = \omega 2^{k+1} + 2(\omega \times \bar{\omega})^{2^p} + \bar{\omega}^{2^{k+1}} - 2$$

$$\omega^{2^{k+1}} + \bar{\omega}^{2^{k+1}} = \omega^{2^{k+1}} + 2(1)^{2^{[k]}} + \bar{\omega}^{2^{k+1}} + 2 = \omega^{2^{k+1}} + \bar{\omega}^{2^{k+1}}$$

Proof. Prove if $2^p - 1|S_{p-2}$ then $2^p - 1$ is prime

Suppose $2^p - 1|S_{p-2}$ Then there must exist $k \in \mathbb{N}$ such that $S_{p-2} = k(2^p - 1)$ By transitivity

$$\omega^{2^{p-2}} + \bar{\omega}^{2^{p-2}} = k(2^p - 1)$$

subtracting $\bar{\omega}^{2^{p-2}}$ from both sides

$$\omega^{2^{p-2}} = k(2^p - 1) - \bar{\omega}^{2^{p-2}}.$$

Multiplying both sides by $\omega^{2^{p-2}}$

$$\omega^{2^{p-1}} = k\omega^{2^{p-2}}(2^p - 1) - 1$$

Squaring both sides Gives us Equation A:

$$\omega^{2^p} = (k\omega^{2^{p-2}}(2^p - 1) - 1)^2$$

Assume on the contrary that $2^p - 1|S_{p-2}$ but $2^p - 1$ is not prime Let q be the smallest prime factor of $2^p - 1, 2 < q \leq \sqrt{2^p - 1}$

Consider the set of congruence classes mod q denoted by

$$Z_q^* = \{\overline{0}, \overline{1}, \overline{2}, \ldots, \overline{q-1}\}.$$

So we have

$$\begin{aligned} \overline{0} &= \{m \in \mathbb{W}, mq|0, q, 2q, 3q, 4q \ldots\} \\ \overline{1} &= \{m \in W, mq+1|1, q+1, 2q+1, 3q+1 \ldots\} \\ \overline{2} &= \{m \in W, mq+2|2, q+2, 2q+2, 3q+2 \ldots\} \\ &\vdots \\ \overline{q-1} &= \{m \in W, mq+(q-1)|q-1, 2q-1, 3q-1, 4q-1 \ldots\} \end{aligned}$$

Let the group X be all possible combinations of a and b such

$$X = \{a, b \in Z_q^* | a + b\sqrt{3}\}$$

We know X is closed under multiplication because the product of two arbitrary elements of X is:

$$a_1, a_2, b_1, b_2 \in Z_q^*, (a_1 + b_1\sqrt{3})(a_2 + b_2\sqrt{3}) = (a_1a_2 + 3b_1b_2) + (a_1b_2 + a_2b_1)\sqrt{3}.$$

$$|X| = q^2$$

Let X^* be the set of all invertable elements of X. Since $(0 + 0\sqrt{(3)})$ is not invertable, it is known

$$|X^*| \leq |X| - 1$$

$$|X^*| \leq q^2 - 1 < q^2$$

It is known $\omega = 2 + \sqrt{(3)} \in X$ Because $q|2^p - 1$, we know $q|k(2^p - 1)$. Therefore,

$$(k(2^p - 1)\omega^{2^{p-2}}) = 0 \text{ in } X.$$

Plugging this into equation A:

$$\omega^{2^p} = (0-1)^2 = 1$$

Therefore,

$$|\omega||2^p \text{but} |\omega| \nmid 2^{p-1}.$$

Since 2^{p-1} and 2^p share all their prime factors, if $|\omega||2^p, |\omega| \nmid 2^{p-1}$ then

$$|\omega| = 2^p.$$

It is known

$$|\omega| \leq |X^*$$

By transitivity we have $2^p < q^2$, which is a contradiction. $\square$

Theorem 8 (Fermat's Little Theorem)**.** *If* p*, a prime* $p \nmid a, a \in \mathbb{N}$*, then* $p | a^{p-1} - 1$

Proof. Let $S = \{a, 2a, 3a, \ldots, (p-2)a, (p-1)a\}$

Because $gcd(a, p) = 1$ and $gcd(k, p) = 1, \forall k < p \in \mathbb{N}$, Thus, $\forall x \in S$, $p \nmid x$

$$|S| = p - 1$$

$$\nexists r, s \in \mathbb{N}, 1 \leq r < s \leq p - 1$$

such $ra \cong sa(modp)$ because there would $\exists s - r \in \mathbb{N}$, $1 \leq s - r \leq p - 1$

such $p|(s - r)a$

Let

$$Z_{p-1*} = \{1, 2, 3, ..., p - 1\}$$

Since $|S| = p - 1$, $\nexists xa \in S$ such $xa \cong 0(modp)$ and no two elements of S are congruent modp,

each element of S will be congruent to one and only one element of Z_{p-1*} and vice versa.

Thus

$$a \times 2a \times 3a \ldots \times (p - 1)a \cong 1 \times 2 \times 3 \times 4 \ldots \times (p - 1)(modp)$$

Giving the result

$$(p - 1)!a^{p-1} \cong (p - 1)!(modp)$$

$$a^{p-1} \cong 1(modp)$$

□

Theorem 9 (Wilson's Theorem)**.** *If* p *is prime, then*

$$(p - 1)! \cong -1(modp)$$

Proof.

$$S = \{2, 3, \ldots, p - 3, p - 2\} \Pi S = (p - 2)!$$

It follows from lagrange's theorem, $\forall a \in S, \exists! b \in S, a \neq b$ such:

$$ab \cong 1(modp).$$

Thus ΠS can be written as the product of $\frac{p-3}{2}$ pairs of $1 \leq i \leq (p - 3)/2$, $a_i b_i \cong 1(modp)$ as follows:

$$\Pi S = (a_1 \times b_1) \times (a_2 \times b_2) \times ... \times (a_{\frac{p-3}{2}} \times b_{\frac{p-3}{2}}) \cong 1 \times 1 \times ... \times 1(modp)$$

Substituting $(p - 2)!$ for ΠS:

$$(p - 2)! \cong 1(modp)$$

Multiplying both sides by (p-1):

$$(p - 1)! \cong -1(modp)$$

□

Theorem 10 (Euler Criterion). *If p is prime, $a \in \mathbb{Z}^+$, $gcd(a,p) = 1$*
Then

$$a^{(p-1)/2} \cong \begin{cases} 1(modp) & if(\frac{a}{p}) = 1 \\ -1(modp) & if(\frac{a}{p}) = -1 \end{cases}$$

Proof. Case One:

$$(\frac{a}{p}) = 1$$

$\exists x \in \mathbb{Z}^+$, $p \nmid x$, $x^2 \cong a(modp)$.
Raising both sides to (p-1):

$$x^{p-1} \cong a^{(p-1)/2}$$

Applying Fermat's little theorem:

$$1 \cong a^{(p-1)/2}$$

Case Two:

$$(\frac{a}{p}) = -1$$

Let $\mathbb{Z}_p = \{1, 2, 3, \ldots, p-2, p-1\}$ Fact: because $gcd(a,p) = 1 \exists! s \in \mathbb{Z}_p \forall x \in \mathbb{Z}_p$ such

$$sx \cong a(modp).$$

Thus, there exists $\frac{p-1}{2}$ unique pairs of m_i, n_i, where $m_i, n_i \cong a(modp) \forall i \in N 1 \leq i \leq \frac{p-1}{2}$.
Therefore,

$$1 \times 2 \times 3 \ldots \times (p-1) = (m_1 \times n_1) \times (m_2 \times n_2) \times \ldots (m_{\frac{p-1}{2}} \times n_{\frac{p-1}{2}})$$

$$(p-1)! \cong a^{(p-1)/2}$$

Applying Wilson's theorem:

$$a^{(p-1)/2} \cong -1(modp)$$

□

Theorem 11. *If $2^p - 1$ is prime, then $2^p - 1 | S_{p-2}$*

Proof. Assume on the contrary there exists a composite $Q, Q = 2^p - 1$ such $S_{p-2} | Q$

Let $\mathbb{Z}_Q^*$ be the set of all congruence classes mod Q

$$\mathbb{Z}_Q^* = \{\bar{0}, \bar{1}, \bar{2}, ...Q \bar{-} 1\}$$

Let X be the group of all possible combinations of $a, b \in \mathbb{Z}_Q^*$ such

$$X = \{a, b \in \mathbb{Z}_Q^* | a + b\sqrt{3}\}$$

Consider $z \in X$, $z = 1 + \sqrt{3}$

Now consider, $z^Q \in X$, when expanded $z^Q = (1+\sqrt{3})^Q = 1^Q + \binom{Q}{Q-1}1^{Q-1}\sqrt(3) + \binom{Q}{Q-2}1^{Q-2}\sqrt{3^2} + \ldots + \binom{Q}{2}1^2\sqrt{3^{Q-2}} + \binom{Q}{1}1^1\sqrt{3^{Q-1}} + \sqrt{3^Q}$

It is known Q divides all but the first and last term of an expanded z^Q, therefore, $z^Q \cong (1+\sqrt{3})^Q \cong 1^Q + \sqrt{3^Q} \cong 1 + \sqrt{3^Q}(modQ)$.

Fact: Because $Q \cong 7(mod12)$, it is known

$$(\frac{3}{Q}) = -1.$$

Therefore by Euler‘s Criterion,

$$3^{(Q-1)/2} \cong -1(modQ).$$

It is known $\sqrt{3^Q} = 3^{(Q-1)/2}\sqrt{3}$,

therefore, $1 + \sqrt{3^Q} \cong 1 + 3^{(Q-1)/2}\sqrt{3} \cong 1 - \sqrt{3}(modQ)$

$$(1+\sqrt{3})^Q \cong 1 - \sqrt{3}$$

Multiplying both sides by $(1+\sqrt{3})$:

$$(1+\sqrt{3})^{Q+1} \cong -2(modQ)$$

Fact: $\omega = 2+\sqrt{3}$, $\bar{\omega} = 2-\sqrt{3}$, $\omega\bar{\omega} = 1$, $2\omega = 4 + 2\sqrt{3} = (1+\sqrt{3})^2$,

$$(1+\sqrt{3})^{Q+1} \cong (2\omega)^{(Q+1)/2} \cong -2(modQ).$$

Fact: By the second supplement to the law of Quadratic Residues,

Since $Q \cong -1(mod8)$,

$$(\frac{2}{Q}) = 1.$$

By Euler‘s Criterion: $2^{(Q-1)/2} \cong -1(modQ)$

$$(2\omega)^{(Q+1)/2} \cong -2(modQ)$$

$$2^{(Q+1)/2}\omega^{(Q+1)/2} = 2 \times 2^{(Q-1)/2}\omega^{(Q+1)/2} \cong -2(modQ)$$

Dividing both sides by 2 and applying Euler‘s Criterion:

$$\omega^{(Q+1)/2} \cong -1(modQ)$$

$Q = 2^p - 1$, $(Q+1)/2 = 2^p/2 = 2^{p-1}$

$$\omega^{2^{p-1}} = \omega^{2^{p-2}}\omega^{2^{p-2}}$$

$$\omega^{(Q+1)/2} = \omega^{2^{p-1}} = \omega^{2^{p-2}}\omega^{2^{p-2}} \cong -1(modQ)$$

Multiplying both sides by $\bar{\omega}^{2^{p-2}}$:

$$\omega^{2^{p-2}}(\omega\bar{\omega})^{2^{p-2}} \cong -\bar{\omega}^{2^{p-2}}$$

Adding $\bar{\omega}^{2^{p-2}}$ to both sides gives us:

$$\omega^{2^{p-2}} + \bar{\omega}^{2^{p-2}} \cong 0(modQ)$$

Thus since $\omega^{2^p} + \bar{\omega}^{2^p} = S_p$

$$Q|S_{p-2}$$

□

Therefore $Q = 2^p - 1$ is prime iff $Q|S_{p-2}$

1.5 Investigation of Abundant Numbers

We investigate the gap between consecutive abundant numbers.

Additionally, we investigate the which abundant numbers x satisfy $\sigma(x) \pm 2 * n$ for some n. One thing that must be pointed out is that for odd abundant numbers, most seem to be of the form $2 * n = 6mod12$.

2 Existence of Odd Perfect Numbers

A key note for the following theorem is that it does NOT prove an odd perfect number exists. Instead, it is simply stating that IF an odd perfect number does exist, then it must be in the form that will be proved below.

Theorem 12. *If an odd perfect number does exist (N), then it can be written in the form $N = p^\alpha Q^2$ where p and Q are in the form 1 (mod 4) and coprime, p is a prime number, and Q and alpha are any integer*

Proof. There are a few facts to state/prove before we can prove the theorem.

Fact 1: An odd number plus an odd number is even (this can easily be proved by writing out the form of an even and odd number)

Fact 2: An odd number plus an even number is odd (can be proved following the same steps as fact 1)

Fact 3: The divisors of p^n are 1, p, p^2 ... p^n because p is prime

Fact 4: The sigma function for sum of divisors is multiplicative: $\sigma(NM) = \sigma(N)\sigma(M)$ where N and M have no prime factors in common

Fact 5: $\sigma(p^n)$ is even if n is odd and is odd if n is even. The sigma function represents the sum of its divisors.

$$\sigma(p^n) = 1 + p + p^2 + ... + p^n$$

If n is odd then there is an even number of terms (remember it starts with 1+...). From fact one, the sum of an even number of odd terms is always even. And from fact 2 the sum of an odd number of odd terms is always odd. Therefore we proved fact five.

Now we can prove the theorem:

By definition, $\sigma(N) = 2N$ where there is only one even factor (which is 2). Write out the prime factorization of N:

$$let\ N\ = p_1^{a_1} p_2^{a_2} ... p_n^{a_n}$$

Using fact four, we can rewrite the equation:

$$2N = \sigma(N) = \sigma(p_1^{a_1} ... p_n^{a_n}) = \sigma(p_1^{a_1}) ... \sigma(p_n^{a_n})$$

Once the equation is expanded, it can be seen that only one of the sigma functions can be even (a factor of 2). The rest of the sigma functions must be odd.

Since from fact five it was proven that $\sigma(p^n)$ is even if n is odd then all a_i in the prime factorization of N is even except for one.

We can therefore create a generalized equation from this fact: $N = p^x Q^2$ where x is an odd integer. Q^2 represents all of the prime factors raised to an even number since Q is simply an integer.

Now we need to prove that p and x must be in the form of 1 (mod 4) or -1 (mod 4). Since the sigma function of N can only have one even factor, $\sigma(p^x)$ can only be divisible by 2 and not 4. Use casework to determine the form of p.
Case 1: let p = -1 (mod 4)

$$\sigma(p^x) = 1 + p^1 + p^2 + ... + p^x \equiv 1 + (-1) + 1 + ... + 1 + (-1) = 0 \pmod 4$$

This is incorrect because it cannot be divisible by 4 as we stated above.
Case 2: let p = 1 (mod 4)

$$\sigma(p^x) = 1 + p^1 + p^2 + ... + p^x \equiv 1 + 1 + ... + 1 + 1 = x + 1 \pmod 4$$

We only proved the form of p, but the same process can be used to determine that x must be in the form 1 (mod 4)

Therefore we proved that $N = p^\alpha Q^2$ where p and Q are in the form 1 (mod 4) and coprime, p is a prime number, and Q and alpha are any integer $\square$

Proof 1: Geometric Series sum
Let

$$S_n = \sum_{i=0}^{n} ar^n$$

$$r \times S_n = r \times (\sum_{i=0}^{n} ar^n)$$

$$r \times S_n = \sum_{i=0}^{n} ar^{n+1}$$

$$r \times S_n = \sum_{i=1}^{n+1} ar^n$$

$$r \times S_n = \sum_{i=0}^{n} ar^n + (ar^{n+1} - a)$$
$$r \times S_n = S_n + a(r^{n+1} - 1)$$
$$(r-1) \times S_n = a(r^{n+1} - 1)$$
$$S_n = \frac{a(r^{n+1} - 1)}{r - 1}$$

2.1 Odd Perfect Number Form

Theorem 13. *An odd perfect cannot be of the form* $6n - 1$

Fact: $\sum_{d|N, d<\sqrt(N)} d + \frac{N}{d} = \sigma(N)$
$N = 6k - 1k \in N$

Then $\sigma(N) = 12k - 2$ Since $N = 6k - 1, N \cong -1(mod3)$ Let d be a divisor of N such $d|N, d \times \frac{N}{d} = N$ Thus it is known $d \times \frac{N}{d} \cong -1(mod3)$ and either $d \cong -1(mod3)\text{and}\frac{N}{d} \cong 1(mod3)\mathbf{OR}d \cong 1\text{and}\frac{N}{d} \cong -1(mod3)$

Either way, it is known $d + \frac{N}{d} \cong 0(mod3)$ And thus $\sum_{d|N,d<\sqrt(N)} d + \frac{N}{d} \cong 0(mod3)$ Therefore $\sigma(N) \cong 0(mod3)$ And Hence N is not perfect because $12k - 2 \not\cong 0(mod3)$

Theorem 14. *An odd perfect number N is of he form* $N = 12k + 1$ *or* $N = 36k + 9$

Proof. Prove: An odd perfect number must be of the form 12k+1 or 36k + 9

Fact: An odd perfect cannot be of the form 6k - 1 Fact: An odd perfect must be of the form 4n + 1

Thus an odd perfect must be either of the form 6k + 1 or 6k + 3

$$\{6k + 1, 6k + 3\} = \{12k + 1, 12k + 3, 12k + 7, 12k + 9\}$$
$$\{4k + 1\} = \{12k + 1, 12k + 5, 12k + 9\}$$

Therefore any odd perfect, N, $N = \{12k + 1, 12k + 9\}$ If $N = 12k + 9, 3 \nmid k\text{then}gcd(3, 4k + 3) = 1$ Therefore

$$\sigma(N) = \sigma(12k+9) = \sigma(3(4k+3)) = \sigma(3)\times\sigma(4k+3) = 4\times\sigma(4k+3), \text{thus} sigma(N) \cong 0(mod4)$$

Since N is perfect $\sigma(N) = 2N = 24k + 18 \not\cong 0(mod4)$ Thus if $12k + 9$ is perfect then $3|k$ Therefore $N = \{12k + 1, 36m + 9\}$ □

We must note that for all odd numbers with positive abundancies, only six numbers of the form 0 mod 12, 36, 1692, 2388, 6552, 7020, and 8496, occur with five of them for a certain number under 10^9 while 6552 occurs for a number around $1.98 * 10^12$. No numbers are known that share the same abundancy that is of the form 0 mod 12 either. For all abundancies between 0 and 100, only three known abundancies occur that are not of the form 6 mod 12, 26 (78975), 36(2205), and 74(1575). 36 is also the smallest known abundancy for an odd number that is a square. Also, the closest abundancy that occurs to 0 for an odd number is six, and so far only three odd numbers are known to have an abundancy of six.

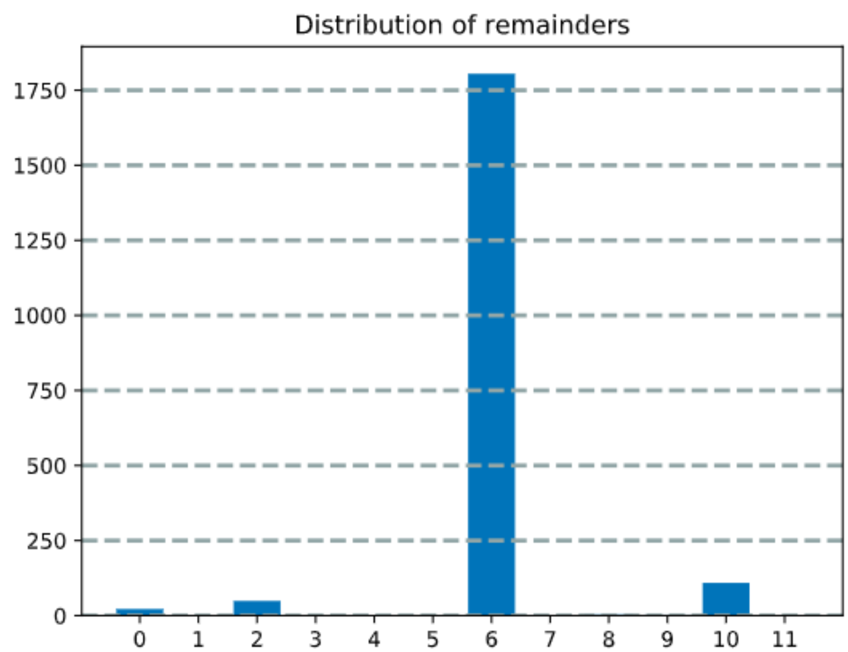

Figure 1: The distribution of abundancies for abundant numbers mod12

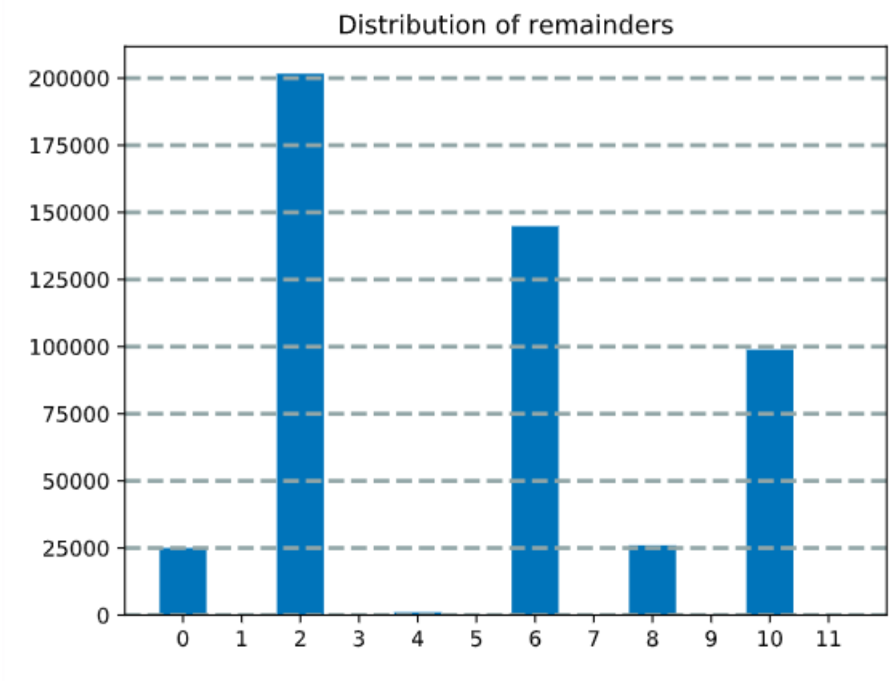

Figure 2: The distribution of deficiencies for deficient numbers mod12

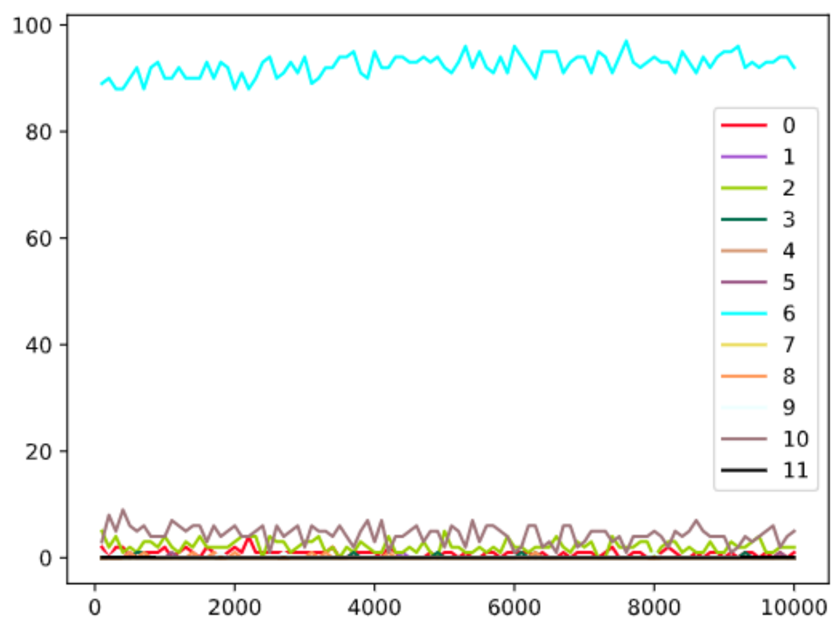

Figure 3: The distribution of abundancies mod12 over multiple ranges

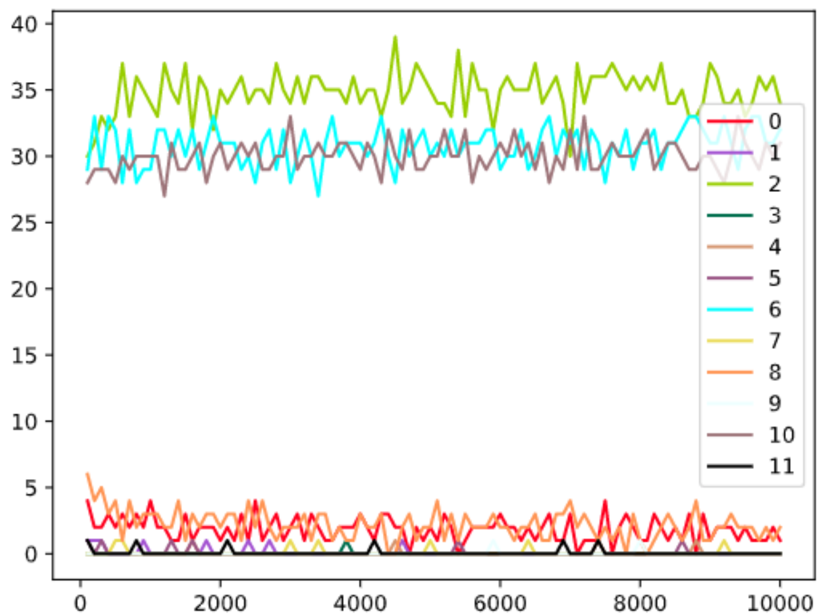

Figure 4: The distribution of deficiencies mod12 over multiple ranges

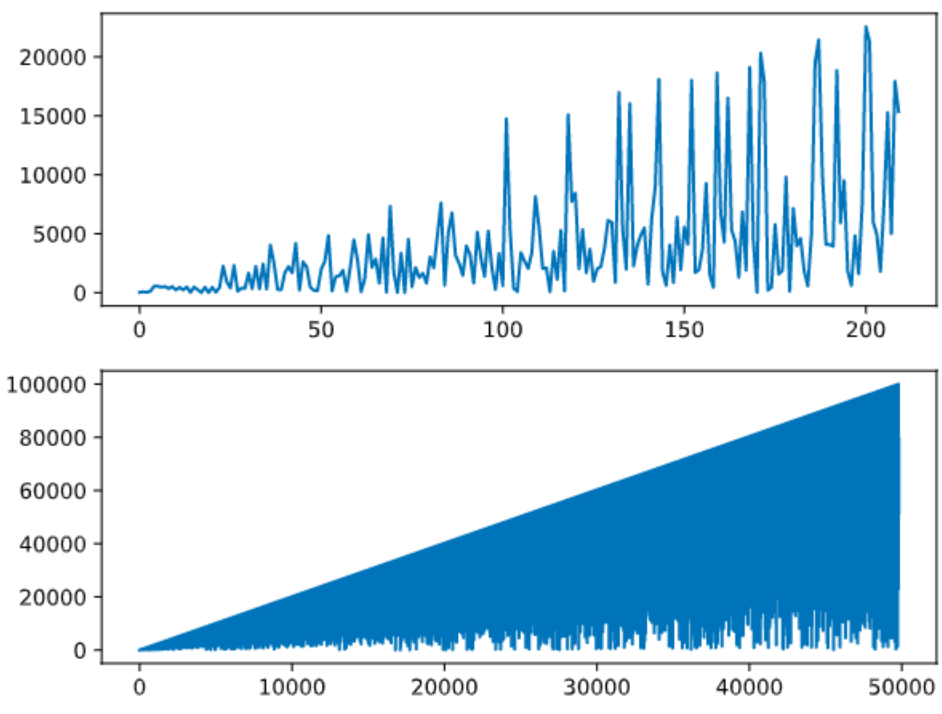

Figure 5: The abundancies/deficiencies for abundant/deficient numbers

3 Additivity of Knot Crossings

3.1 Abstract

The conjecture that the crossing number of a knot is additive under connected sum has been an open problem for over 100 years. In 1987, Louis Kauffman proved that the crossing number of alternating links is additive. More recently, in 2003, Yuanan Diao proved that the crossing number is additive for torus links where $Cr\left(T_1 \sharp T_2 ... \sharp T_m\right) = Cr\left(T_1\right) + Cr\left(T_2\right) + ... + Cr\left(T_m\right)$. To better understand this problem, we will introduce foundational topics in topology and go in depth on different knot invariants such as Alexander and Jones Polynomial to help us classify knots.

3.2 Introduction

Definition 4. A knot is a closed loop in three dimension. There are many different kinds of knots. The simplest type is called the unknot. An example of an unknot is a rubber band where there are no crossings in it. Knot diagrams are a projection of knots onto a plane in order to better visualize and manipulate them.

Definition 5. A crossing number of a knot is the least number of crossings in its knot diagram.

Definition 6. A connected sum of two knots can be done by cutting each one open and then joining them together through a straight bar.

Conjecture 3. *The crossing number of a knot is additive under connected sum*

3.3 Euler Characteristic

Definition 7. Let Γ be a graph. The Euler Characteristic of Γ is the number $\chi(\Gamma) = V - E + F$ where V represents the number of vertices, E represents the number of edges, and F represents the number of faces. The number of faces include the unbounded face outside of the graph, which is considered in "infinite space".

3.4 Planar Graphs

Definition 8. A *planar graph* is a 2–dimensional graph where no two edges cross one another.

Theorem 15. *For any finite, connected, planar graph* Γ, $\chi(\Gamma) = V - E + F = 2$

Proof. We will prove it using induction on E, the number of edges:

Base Case:

When E = 0 that means there has to be a single vertex and therefore only one face (in the infinite space).

$$\chi = 1 - 0 + 1 = 2$$

Induction Hypothesis:
Assume $\chi = V - E + F = 2$ holds true for all E

Induction Step:
We want to prove that the claim holds true when an extra edge is added
There are two ways to add another edge E_1:

1. Add another vertex:
   By including another vertex, you must connect it to the original graph using edge E_1. The number of faces stays the same. The new equation will become:

   $$\chi = (V + 1) - (E + 1) + F = V - E + F$$

   Using the induction hypothesis, we know that $\chi = V - E + F = 2$

2. Create a loop:
   In order to create a loop, the edge E_1 must start and end on the same vertex. By creating a loop, it must increase the number of faces by one. The number of vertices stays the same. The new equation will become:

   $$\chi = V - (E + 1) + (F + 1) = V - E + F$$

   Using the induction hypothesis, we know that $\chi = V - E + F = 2$
   We have proved that for any finite, connected planar graph

   $$\chi = V - E + F = 2$$

□

3.5 Polyhedrons

An interesting connection is that the Euler Characteristic property for planar graphs works with simple, convex polyhedrons. We will prove this is true using Cauchy's Proof

Definition 9. A *simple polyhedron* is a genus 0 polyhedron.

Proof. Take an arbitrary simple, convex polyhedron and remove one of its faces. You can then proceed to "pull" the graph apart as stated and create a planar graph corresponding to the polyhedron. Since the resultant graph is planar, it must have an Euler Characteristic of 2. Removing a face does not alter the result because it corresponds to the empty space around the planar graph. An example of this procedure is in Figure 6. □

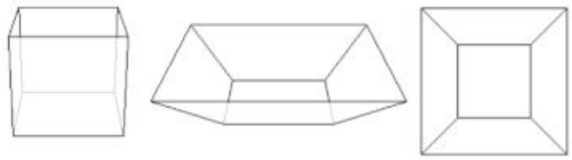

Figure 6: Example of Cauchy's Proof on a cube

Definition 10. A topological invariant is a property that does not change (invariant) under continuous functions (homeomorphisms).

The Euler Characteristic is a topological invariant because if two objects are topologically the same, then they will have the same Euler Characteristic. Yet an important distinction is that the converse is not necessarily true (i.e if two objects have the same Euler Characteristic it does not necessarily mean they are topologically same)

Definition 11. Two objects are considered homeomorphic if one can be distorted into another.

For example, all simple convex polyhedrons are homeomorphic to a sphere. To do this, imagine "inflating" the polyhedron until they are round like a sphere. This therefore means that all spheres also have a Euler characteristic of 2.

3.6 Genus

Definition 12. The *genus* of a surface is the number of "holes" that a surface has.

Before we found the Euler Characteristic of simplex convex polyhedrons. Now we will find the Euler Characteristic of polyhedrons that have a genus.

Theorem 16. *For a closed orientable surface,* $\chi = 2 - 2g$

Proof. We will first prove with a specific case. Take a polyhedron and create one genus (like drilling a hole) by removing a top and bottom face. In order to connect the hole, add new faces and edges. For example, if the top face that was removed is a n-gon, then add n faces and n edges in order to connect the hole. We proved before that the Euler Characteristic for all simple convex polyhedron is $\chi = V - E + F = 2$. Yet after creating a genus to the polyhedron, n edges and n faces are added to connect the hole and 2 faces are removed from creating a genus. When the genus is 1, the new equation is:

$$\chi = V - (E + n) + (F + n - 2) = V - E + F - 2 = 0$$

For the more generalized case, follow the same procedure and instead create more "holes" in order to satisfy the amount of genus. The formula is $\chi = 2 - 2g$ because every time a new hole is created, two faces are removed and therefore 2g is subtracted from the Euler Characteristic of a simple convex polyhedron. □

Definition 13. A *Seifert surface* of a knot K is a surface whose boundary is K. A knot infinitely many Seifert surfaces.

In order to create a Seifert surface from a knot, first assign an orientation to the knot and then create Seifert circles by getting rid of the crossing. To guarantee that the circles don't intersect one another, assign them to different heights. Lastly, connect the Seifert circles using twisted bands where the crossings were supposed to be. Look at Figure 7 for an example of the procedure to make a trefoil knot into its Seifert surface.

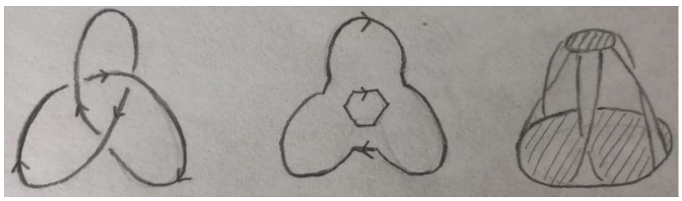

Figure 7: Formation of a Seifert Surface

Definition 14. The genus of a knot is the minimal genus of all Seifert surfaces bounded by the knot

Definition 15. The boundary component is the amount of continuous surfaces on the boundary of an object. Therefore as more faces are removed from the boundary, the boundary component increases. The boundary component of a knot is always one.

The introduction to boundary components will allow us to expand the formula for Euler Characteristic in terms of the genus. $\chi(S) = 2 - 2g(S) - \mu(S)$, where S is the Seifert Surface.

Theorem 17. $g = \frac{C-S+1}{2}$ *where* g = *genus of a Seifert surface,* C = *number of crossings on the knot (the boundary of the Seifert surface),* S = *number of Seifert circles*

Proof. To prove this theorem, we will investigate the properties of the twisted bands that connect the Seifert Surfaces.

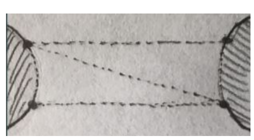

Figure 8: Twisted Bands of Seifert Surfaces

We can conclude that there are C total bands because C represents the total number of crossings which are replaced by the twisted bands. There is also a total of 4C vertices because each band can be labelled as a rectangle with 4 vertices. By splitting the face into 2 triangles, there is a total of 2 faces for each band. There is also S faces in total because we need to count the faces formed by the Seifert circles. Therefore there are 2C+S faces. From the rectangle and edge to create the two triangles, there is a total of 5 edges for each band. Yet there's also 2 more edges for each band from the Seifert circle. In total there are 7C faces.

Using previously proven formulas for the Euler Characteristic, we can now substitute with the new values found from the bands. C bands

$$\chi = V - E + F$$

$$= 4C - 7C + (2C + S)$$

$$\chi = S - C$$

$$\chi(S) = 2 - 2g(S) - \mu(S)$$

The boundary of a Seifert surface is a knot, so $\mu(S)$ can be rewritten as $\mu(K)$ which is 1 by definition

$$2g = 2 - \chi - \mu(K)$$

$$g = \frac{2 - (\chi + \mu(K))}{2}$$

$$= \frac{2 - (S - C + \mu(K))}{2}$$

$$= \frac{2 - (S - C + 1)}{2}$$

$$g = \frac{C - S + 1}{2}$$

□

Theorem 18. *The genus number is additive when taking connected sum of knots*

Proof. Prove that $g(K \sharp J) = g(K) + g(J)$ where K and J are knots.

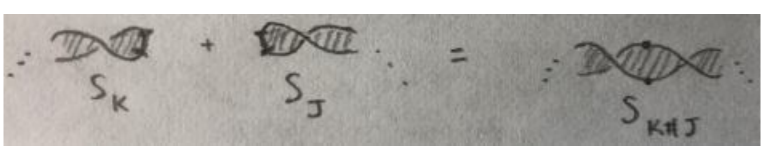

Figure 9: Connected Sum of Seifert Surfaces

Use the Euler Characteristic property on the Seifert surfaces of knots K and J where S_K is a Seifert surface with knot K as the boundary and S_J is a Seifert surface with knot J as the boundary. Since $g(K) = g(S_K)$, $g(J) = g(S_J)$, we

can use Seifert surfaces to prove this theorem.
Let S_K have V_1 vertices, E_1 edges, and F_1 faces. Let S_J have V_2 vertices, E_2 edges, and F_2 faces.

$$\begin{aligned}\chi\left(S_{K\sharp J}\right) &= \left(V_1+V_2-2\right)-\left(E_1+E_2-1\right)+\left(F_1+F_2-1\right)\\ &= V_1-E_1+F_1+V_2-E_2+F_2-2\\ &= \chi\left(S_K\right)+\chi\left(S_J\right)-2\end{aligned}$$

From Theorem 5, we know:

$$\chi\left(S_{K\sharp J}\right)=2-2g\left(K\sharp J\right),\ \chi\left(S_K\right)=2-2g\left(K\right),\ \chi\left(S_J\right)=2-2g\left(J\right)$$

Substituting these values to the previous formula:

$$\begin{aligned}2-2g\left(K\sharp J\right) &= 2-2g\left(K\right)+2-2g\left(J\right)-2\\ 1-g\left(K\sharp J\right) &= 1-g\left(K\right)+1-g\left(J\right)-1\\ g\left(K\sharp J\right) &= g\left(K\right)+g\left(J\right)\end{aligned}$$

□

4 Knot Invariant

Definition 16. A knot invariant is a quantity that is the same for all equivalent knots.

This is extremely useful because it can help determine whether a knot is the same or different from another. Yet a key distinction, just like topological invariants, is that if two knots have the same knot invariant doesn't always mean they are exactly same. But if two knots have different knot invariants then they are definitely different knots. There are many different types of knot invariants. We explore a couple of them below.

4.1 Tricolorability

A knot is either colorable or uncolorable. In order to be colorable, it must satisfy these rules: At least two colors must be used and at each crossing it must either be all the same color or all different colors.
For example an unknot is not colorable because it only uses one color, while a trefoil is colorable because it uses three colors and satisfies the rules at each crossing.

Theorem 19. *Tricolorability is a knot invariant*

Proof. We will prove this is true using Reidemeister moves.

Definition 17. Reidemeister moves are three different moves on a knot diagram that will guarantee they are the same knot. Type 1 is to twist and untwist in any direction. Type 2 is to move one loop over another. Type 3 is to move a string over or under a crossing.

Since any move from one knot diagram to another is through a series of Reidemeister moves, if we can prove that each Reidemeister move preserves colorability, then the final knot diagram will still be tricolorable. This can easily be proved by drawing out pictures for each Reidemeister move and showing that it satisfies the colorability rules. ☐

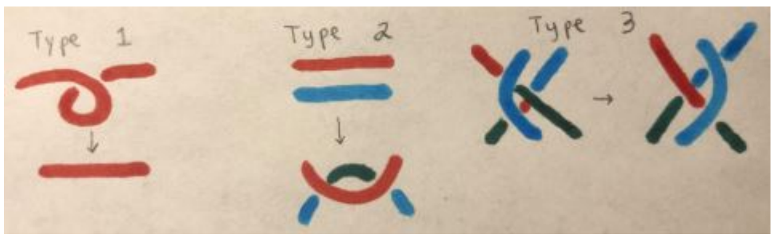

Figure 10: Reidemeister Moves Satisfying Colorability Rules

This theorem is extremely useful. For example, since tricolorability is a knot invariant, the trefoil knot can never be manipulated by Reidemeister moves to become non-tricolorable. The trefoil knot, then, must be distinct from the unknot.

4.2 Knot Determinant

Another useful knot invariant is the knot determinant which assigns knots to an integer based on the general steps explained below. This property will be a foundation for other knot invariants such as polynomials. We will not prove why knot determinants is an invariant and rather show an example (using a trefoil knot) on how this property works.

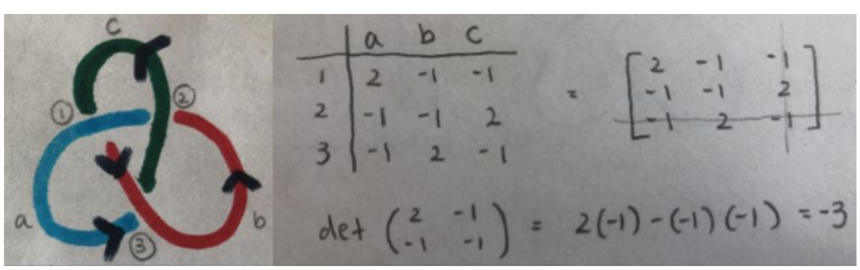

Figure 11: Knot Determinant of a Trefoil Knot

Rules: Label the trefoil knot where the numbers represent the crossings and the alphabet represent the arcs. Then form a matrix of the knot where each row labels a crossing and each column labels an arc. Within the matrix, the numbers 0, -1 and 2 will represent the type of crossing. Let 0 be an arc not involved in a crossing, -1 be an undercrossing and 2 be an overcrossing. Find the determinant of the matrix by removing any row and column. The prime factors of the determinant is the number of ways a knot can be colored.

4.3 Alexander Polynomial

Using the information from knot determinants, Alexander Polynomial, $\Delta(t)$, generalizes the results. Instead of using integers, this property uses polynomials to label the knot diagrams. There are a few key distinctions:

1. Rather than labelling the crossings as over and undercrossings, the crossings can be determined using right handedness or left handness. Place your thumb in the direction of the upper strand and your fingers in the direction of the bottom strands. After matching the orientation of the crossings, whichever hand you use will determine if it's a right handed or left handed crossing. The strand on top is always 1-t while the bottom ones are labelled t and -1 based on which ones on the left or right. Therefore in the matrix the values used are 1-t, t, -1 instead of 0,-1,2.

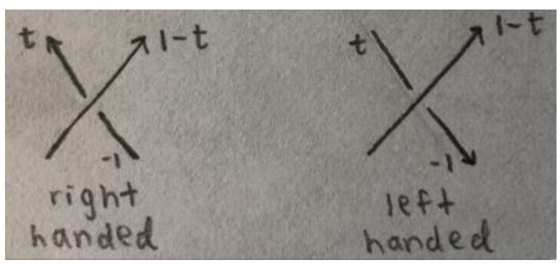

Figure 12: Determine Crossings for Alexander Polynomial

2. When taking the determinant of the resulting matrix, it is important to realize that the Alexander polynomial is invariant up to $\pm t^m$ where m is an integer. This means that if two Alexander polynomials differ by only a factor of $\pm t^m$ it does not necessarily mean they are different knot diagrams. Yet having the same Alexander polynomial does not guarantee it is the same knot.

There is also a direct connection with knot determinants.

Theorem 20. $\Delta_K(-1) = \det(K)$

Proof. When replacing t with -1, the top strand becomes $1 - t = 1 - (-1) = 2$ and the bottom strand becomes -1 (original was t and -1). This satisfies the rule for crossings when determining knot determinants because it was defined that

-1 was an undercrossing and 2 was an overcrossing. This proves the connection between Alexander Polynomial and knot determinants. □

Below is an example of finding the Alexander Polynomial of a trefoil knot:

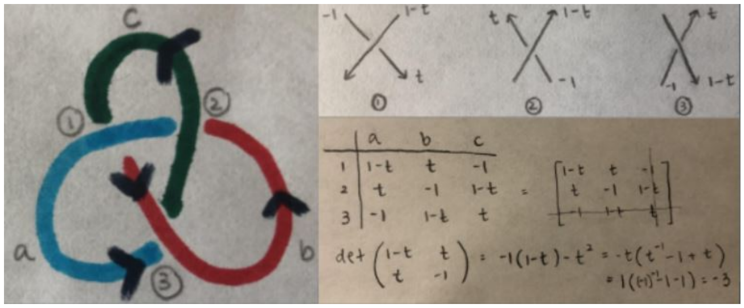

Figure 13: Alexander Polynomial of Trefoil Knot

4.4 Jones Polynomial

Another more powerful polynomial is the Jones Polynomial which is also a knot invariant. This polynomial provides connections between different knots and introduces the idea of links.

Definition 18. A link uses more than one string (otherwise known as component). A knot is a link with one component.

Instead of classifying crossings based on right/left handness or over/undercrossings, another way is through the Skein relation.

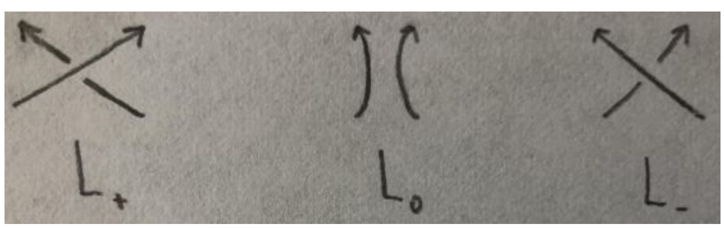

Figure 14: Skein Relation

The Jones Polynomial satisfy the following two rules:

$$V(O) = 1 \, where \, O \, is \, an \, unlink$$

$$t^{-1} V(L_+) - tV(L_-) + \left(t^{-\frac{1}{2}} - t^{\frac{1}{2}}\right) V(L_0) = 0$$

The proof for the rules above is a bit complicated so we will only provide examples on how to use the Jones Polynomial to classify links and knots. For the examples below, our end goal is to determine the Jones Polynomial for the trefoil knot.

The general rule for determining the Jones Polynomial is to define different knots as $V(L_+)$, $V(L_-)$, $V(L_0)$. This can be done by changing the crossings for the original link that is being solved for.

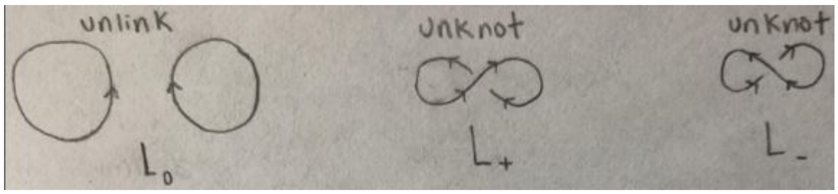

Figure 15: Diagram for Jones Polynomial of Unlink

Example 1 (Jones polynomial for Unlink)**.**

$$t^{-1}V(L_+) - tV(L_-) + \left(t^{-\frac{1}{2}} - t^{\frac{1}{2}}\right)V(L_0) = 0$$

$$t^{-1}(1) - t(1) + \left(t^{-\frac{1}{2}} - t^{\frac{1}{2}}\right)V_U = 0$$

$$V_U = -\frac{t^{-1} - t}{t^{-\frac{1}{2}} - t^{\frac{1}{2}}}$$

$$= -\frac{\left(t^{-\frac{1}{2}} - t^{\frac{1}{2}}\right)\left(t^{-\frac{1}{2}} + t^{\frac{1}{2}}\right)}{t^{-\frac{1}{2}} - t^{\frac{1}{2}}}$$

$$V_U = -\left(t^{-\frac{1}{2}} + t^{\frac{1}{2}}\right)$$

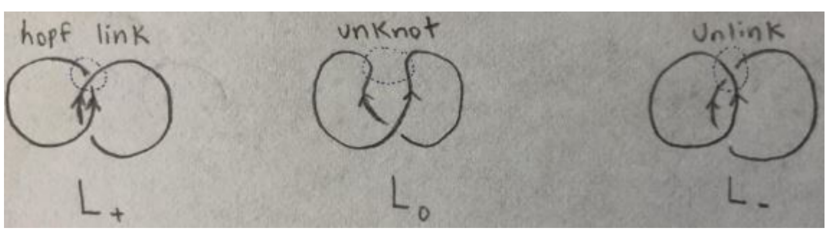

Figure 16: Diagram for Jones Polynomial of Hopf Link

Example 2 (Jones polynomial for Hopf Link)**.**

$$t^{-1}V(L_+) - tV(L_-) + \left(t^{-\frac{1}{2}} - t^{\frac{1}{2}}\right)V(L_0) = 0$$

$$t^{-1}V_H - t\left(-\left(t^{-\frac{1}{2}} + t^{\frac{1}{2}}\right)\right) + \left(t^{-\frac{1}{2}} - t^{\frac{1}{2}}\right)(1) = 0$$

$$t^{-1}V_H + t^{\frac{1}{2}} + t^{\frac{3}{2}} + t^{-\frac{1}{2}} - t^{\frac{1}{2}} = 0$$
$$V_H = -\frac{t^{-\frac{1}{2}} - t^{\frac{3}{2}}}{t^{-1}}$$
$$V_H = -t^{\frac{5}{2}} - t^{\frac{1}{2}}$$

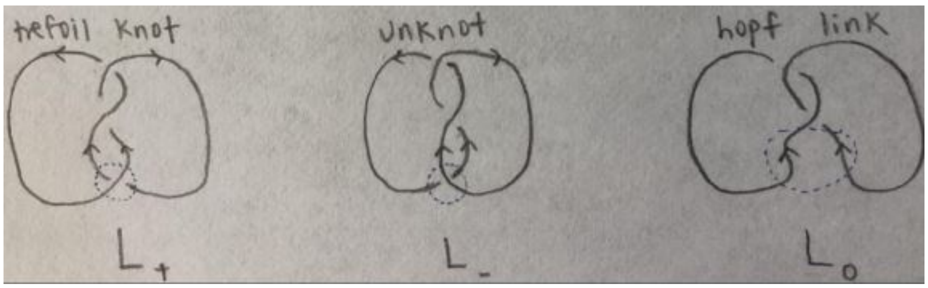

Figure 17: Diagram for Jones Polynomial of Trefoil Knot

Example 3 (Jones polynomial for Trefoil Knot)**.**

$$t^{-1}V(L_+) - tV(L_-) + \left(t^{-\frac{1}{2}} - t^{\frac{1}{2}}\right)V(L_0) = 0$$
$$t^{-1}V_T - t(1) + \left(t^{-\frac{1}{2}} - t^{\frac{1}{2}}\right)\left(-t^{\frac{5}{2}} - t^{\frac{1}{2}}\right) = 0$$
$$V_T = t^2 + t\left(t^2 + 1 - t^3 - t\right)$$
$$V_T = -t^4 + t^3 + t$$

Based on the examples above, a generalized formula can be created for the Jones Polynomial of a m-component unlink:

$$V(m - component\, unlink) = (-1)^{m-1}\left(t^{-\frac{1}{2}} + t^{\frac{1}{2}}\right)^{m-1}$$

When t=1

$$t^{-1}V(L_+) - tV(L_-) + \left(t^{-\frac{1}{2}} - t^{\frac{1}{2}}\right)V(L_0) = 0$$
$$V_{L+}(1) - V_{L-}(1) + 0 = 0$$
$$V_{L+}(1) = V_{L-}(1)$$

From the formula above, we know that the Jones Polynomial of any link when t=1 does not change when changing crossings. Therefore the Jones Polynomial of any link at 1 should be the same as the Jones Polynomial of the trivial link at 1.

$$V_L(1) = V_{trivial\, link}(1)$$
$$= (-1)^{m-1}\left(2^{m-1}\right)$$
$$V_L(1) = (-2)^{m-1}$$

From this formula we know that all knots must have a Jones Polynomial of 1 when t=1 because a knot only has one component.